Von Nemann envelope of a C^*-algebra.

Now we are going to find a functor

$$vN : \mathbf{C^*\text{-}Alg} \longrightarrow \mathbf{vN\text{-}Alg}$$

which is left adjoint of forgetting of the predual

$$\mathbf{C^*\text{-}Alg} \longleftarrow \mathbf{vN\text{-}Alg}.$$

This means that for evry C^*-algebra B and a von Neumann algebra A

$$\mathbf{C^*\text{-}Alg}(B, A) = \mathbf{vN\text{-}Alg}(vN(B), A)$$

In other words, we will show that for every C^*-algebra B we can construct a von Neumann algebra $vN(B)$ and a $*$-algebra map $\rho : B \to vN(B)$ satisfying the following universal property: for every von Neumann algebra A, every $*$-algebra map $\phi : B \to A$ admits a unique factorization

$$\begin{tikzcd} B \arrow[r, "\forall \phi"] \arrow[d] & A \\ vN(B) \arrow[ur, dashed, "\exists ! \psi"'] & \end{tikzcd}$$

where ψ is an ultraweakly continuous $*$-algebra map of von Neumann algebras.

Then $\rho : B \to vN(B)$ is determined uniquely up to an isomorphism by B. One refers to it as the <u>von Neumann envelope</u> of B.

To verify that $\rho : B \to vN(B)$ is a von Neumann envelope it will suffice to verify the following properties:

(a) The image of ρ is ultraweakly dense in $vN(B)$

(b) Every $*$-algebra map $B \to B(H)$ extends to an ultraweakly continuous $*$-algebra map.
$vN(B) \to B(H)$.

Indeed, if (a) and (b) are satisfied, and there is a $*$-algebra map $\varphi : B \to A$, where $A \subset B(H)$ is a von Neumann algebra, then

(b) $\Rightarrow$ φ extends to $\psi : vN(B) \to B(H)$.

$A \subset B(H)$ ultraweakly closed $\Rightarrow$ $\psi^{-1}(A) \subseteq vN(B)$ ultraweakly closed and $\varphi(B) \subset \psi^{-1}(A)$.

(a) $\Rightarrow$ $\psi^{-1}(A) = vN(B)$ $\Rightarrow$ $\psi : vN(B) \to A$, hence the extension ψ does exist

(a) $\Rightarrow$ (by a continuity argument) ψ is unique.

To construct $vN(B)$, let $S(B)$ the set of states of B.

$\forall \mu \in S(B)$ $\quad H_\mu$ a GNS-construction

i.e. completion of B with respect to (possibly degenerate) scalar product $\langle b', b \rangle := \mu(b^* b')$.

$$H := \bigoplus_{\mu \in S(B)} H_\mu \quad , \qquad vN(B) := \text{ultraweak closure of } B \text{ in } B(H).$$

Hilbert orthogonal sum

$\Rightarrow$ $vN(B)$ is a von Neumann algebra and the image of $B \to vN(B)$ ultraweakly dense

To verify (b) one must show that every representation K of B extends to a von Neumann representation of $vN(B)$.

<u>Fact</u>. Every representation of B can be obtained as a direct sum of cyclic representations.
W.L.O.G. can assume that K is cyclic.

$\Rightarrow \exists\, \mu \in S(B)$ $\quad K = H_\mu$ and then it is obvious that it extends to a von Neumann representation of the ultraweak closure $vN(B)$.

Explicit description of $vN(B)$.

$B \subset B(H)$ is a norm closed $*$-subalgebra and $vN(B)$ is a von Neumann algebra

$\Rightarrow$ we constructed a Banach space D and an isometry $vN(B) \simeq D^*$ carrying the ultraweak topology on $vN(B)$ into the weak*-topology on D^*. Now, $\rho : B \to vN(B) \cong D^*$ is adjoint to a bounded operator $\rho' : D \to B^*$

Lemma. $\rho' : D \to B^*$ is an isomorphism of Banach spaces, i.e. it admits a continuous inverse.

Proof. Open mapping theorem $\Rightarrow$ it is sufficient to prove that ρ' is an algebraic linear isomorphism.

Regard elements of D as ultraweakly continuous functionals $vN(B) \longrightarrow \mathbb{C}$.

B ultraweakly dense in $vN(B) \Rightarrow$ every such a functional is determined uniquely by its restriction to B

$\Rightarrow \rho'$ injective.

To prove surjectivity, we must to extend every continuous functional $\mu : B \to \mathbb{C}$ to an ultraweakly continuous functional $vN(\mu) : vN(B) \to \mathbb{C}$.

It is enough to consider μ s.t. $\mu(b^*) = \overline{\mu(b)}$,
(since B is a complexification of the real Banach space of such functionals)

First, for μ positive: $b \in B_+ \Rightarrow \mu(b) \geq 0$.

$\mu \neq 0 \Rightarrow \frac{\mu}{\|\mu\|} \in S(B)$ and we use the following

Fact. Every state of B extends to an ultraweakly continuous functional on $vN(B)$.
(by construction of $vN(B)$).

To complete the proof of the latter lemma we need another one.

Lemma. Let B be a C^*-algebra, $\mu : B \to \mathbb{C}$ a functional s.t. $\mu(b^*) = \overline{\mu(b)}$. Then there exist positive functionals μ_+, μ_- s.t. $\mu = \mu_+ - \mu_-$ and $\|\mu_+\| + \|\mu_-\| \leq \|\mu\|$.

Proof. W.L.O.G. $\|\mu\| \leq 1$. We want to find $\nu_+, \nu_- \in S(B)$ s.t. $\mu = (1-t)\nu_+ + t(-\nu_-)$.

But $S(B) = \{\nu : A \xrightarrow[\text{linear}]{} \mathbb{C} \mid \nu(b^*) = \overline{\nu(b)}, \|\nu\| \leq 1, \nu(1) = 1\}$

$\Rightarrow S(B)$ closed convex subset of the unit ball in A^*.

Define $S'(B) =$ convex hull of $S(B) \cup -S(B)$.

The unit ball of B^* is weak* compact

$\Rightarrow$ $S(B)$ is weak*-compact $\Rightarrow$ $S'(B)$ weak* compact

$\Rightarrow$ $S'(B)$ weak*-closed in B^*.

It is enough to prove that $\mu \in S'(B)$, or equivalently

$\mu \in$ weak closure of $S'(A)$.

Suppose the opposite. Then there exists a finite sequence of elements of $B_{\mathbb{R}}$ giving a map

$$q : B^*_{\mathbb{R}} \longrightarrow \mathbb{R}^n$$

such that $q(\mu) \notin q(S'(B))$.

But $q(S'(B))$ is closed, convex subset in $\mathbb{R}^n$

$\Rightarrow$ $\exists$ hyperplane separating $q(S'(B))$ from $q(\mu)$.

$\Rightarrow$ $\exists$ $b = b^* \in B$, $\lambda \in \mathbb{R}$

$$\mu(b) > \lambda \quad \& \quad \forall_{\nu \in S'(B)} \quad \nu(b) \leq \lambda$$

$0 \in S'(B) \Rightarrow \lambda \geq 0$

$\nu \in S(B) \Rightarrow \nu(b), -\nu(b) \leq \lambda \Rightarrow \|b\| \leq \lambda$

$\Rightarrow \|\mu(b)\| \leq \underbrace{\|\mu\|}_{\leq 1} \underbrace{\|b\|}_{\leq \lambda} \leq \lambda$. $\lightning$

$\Rightarrow \mu \in S'(B) \Rightarrow \exists\, t \in [0,1],\ \nu_+, \nu_- \in S(B)$ $\mu = (1-t)\nu_+ + t(-\nu_-)$.

$\square$

Now we can continue the discussion of $vN(B)$.

$vN(B) \cong D^*$, $D \cong B^*$ $\Rightarrow$ $\bar{\rho} : B^{**} \cong D^* \cong vN(B)$.

By construction, we have a commutative diagram

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr, "\rho"] & \\ B^{**} \arrow[rr, "\bar{\rho}"] & & vN(B) \end{tikzcd},$$

and $\bar{\rho}^{-1}$ carries the ultraweak topology on $vN(B)$ to the weak* topology on B^{**}.

Fact. In fact $\bar{\rho}$ is an isometry (by construction $\|\bar{\rho}\| \leq 1$ and by the previous Lemma $\bar{\rho}|_{\text{selfadjoint elts of } B}$ is an isometry.

Now we can rewrite the adjunction using the equivalence

$$\mathbf{vN\text{-}Alg}^{op} \simeq \mathbf{Meas\text{-}Coalg}$$

$$A \longmapsto A_*$$

$$C^* \longleftarrow C$$

and the above description of $vN(B) = D^*$ and $vN(B)_* = D = B^*$

$$\mathbf{C^*\text{-}Alg}^{op}(C^*, B) = \mathbf{Meas.Coalg}(C, vN(B)_*)$$

which means that we have a pair of adjoint functors

$$\mathbf{Meas.Coalg} \rightleftarrows \mathbf{C^*\text{-}Alg}^{op}$$

$$C \longmapsto C^*$$
$$vN(B)_* \longleftarrow\!\shortmid\; B$$

Note the analogy with the purely algebraic case, when we have an adjunction

$$\mathbf{Alg}^{op}(C^*, B) = \mathbf{Coalg}(C, M(B, \mathbb{C}))$$

with the following pair of adjoint functors

$$\begin{array}{ccc} \mathbf{Coalg} & \rightleftarrows & \mathbf{Alg}^{op} \\ C & \longmapsto & C^* \\ M(B, \mathbb{C}) & \longleftarrow\!\shortmid & B \end{array}$$

So we have the following analogy between the right adjoints

$$\mathbf{Meas.Coalg} \xleftarrow{vN(-)_*} \mathbf{C^*\text{-}Alg}^{op}$$

$$\mathbf{Coalg} \xleftarrow{M(-,\mathbb{C})} \mathbf{Alg}^{op}$$

Remark. Note that in the algebraic context a more general construction is available $M(B, A)$ for A not necessarily being $\mathbb{C}$. This leads to an enrichment of **Alg** in **Coalg**. Therefore the following problem arises.

Problem. Does the construction $vN(-)_*$ extend to some $vN(-,A)_*$ such that $vN(-,\mathbb{C})_* = vN(-)_*$, we have an adjunction

$$\mathbf{C^*\text{-}Alg}^{op}(\mathbf{Ban}(C,A), B) = \mathbf{Meas\text{-}Coalg}(C, vN(B,A)_*),$$

and an enrichment of $\mathbf{C^*\text{-}Alg}$ in $\mathbf{Meas\text{-}Coalg}$ together with an embedding

$$\mathcal{M}(\mathbf{C^*\text{-}Alg}(B,A)) \hookrightarrow vN(B,A)_* ?$$

where $\mathcal{M}(-)$ denotes some space of measures (regular on a Borel measurable space?)

Exercise 18. Let C be a $*$-coalgebra, A a $*$-algebra. Show that $\mathbf{Vect}(C,A)$ is a $*$-algebra.

Solution. $\varphi : C \to A$, $(\varphi^*)(c) := \varphi(c^*)^*$

$$(\varphi^{**})(c) = (\varphi^*)(c^*)^* = \varphi(c^{**})^{**} = \varphi(c)$$

$z \in \mathbb{C}$ $\quad (z\varphi)(c) := \varphi(cz)$

$$(z\varphi)^*(c) = (z\varphi)(c^*)^* = \varphi(c^*z)^* = (z\varphi(c^*))^* = \bar{z}\varphi(c^*)^*$$

$$= (\bar{z}\varphi^*)(c). \quad \blacksquare$$